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COMMENT

The soliton number of optical soliton bound states for two special families of input pulses

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Abstract. To gain a better understanding of the generation of optical solitons we investigate the linear eigenvalue problem associated with the non-linear Schrödinger equation. Two families of initial envelope functions are discussed. We find that, for a purely imaginary initial envelope function of width a and height b, and its Galilei transforms, the soliton number of soliton bound states is the integer smaller than $\frac{1}{2}+ab/\pi$. For the initial envelope function $\mathrm{i}\beta\exp(-\alpha|x|)$ and its Galilei transforms, the soliton number of soliton bound states is equal to the number of intersections of the Bessel functions $J_{-1/2}$ and $\pm J_{1/2}$ below β/α, which is the integer smaller than $\frac{1}{2}+2\beta/\alpha\pi$.

Soliton propagation in optical fibres was predicted by Hasegawa and Tappert [1] and was first observed by Mollenauer *et al* [2]. Since solitons could play an important role in data communication a good understanding of the soliton content of pulses from different types of sources is important. This understanding can be gained by investigating the linear eigenvalue problem associated with the non-linear Schrödinger equation [3]. This eigenvalue problem is an example of the AKNS 2×2 system and, as such, has been extensively studied [4]. In this respect the situation is similar to the case of the Korteweg-de Vries soliton where the soliton content of a wave is given by the discrete spectrum of the associated Schrödinger equation (for a review, see [5]).

In another aspect, however, the situation is different. Whereas our intuition, trained in quantum mechanics and backed up by sophisticated mathematics, helps us to understand the soliton content of a Korteweg-de Vries wave, comparatively few examples of special input pulses have been worked out in detail to develop our understanding in the case of the eigenvalue problem associated with the non-linear Schrödinger equation. After the pioneering work by Zakharov and Shabat [3], Satsuma and Yajima [6] have started a systematic study of this eigenvalue problem and have discussed the initial envelope function $\gamma\,\mathrm{sech}(x)$ as a special case. Now that the principles of soliton generation are well established experimentally the systematic theoretical study should be intensified. With this perspective in mind we want to add to the understanding of the eigenvalue problem.

The electic field $E(r, \bar{x}, \bar{t})$ in a monomode optical fibre may be expressed as

$$E(r, \bar{x}, \bar{t}) = \mathrm{Re}\{\Phi(\bar{x}, \bar{t})R(r)\exp[\mathrm{i}(k\bar{x}-\omega\bar{t})]\} \tag{1}$$

where k is the wavenumber in the $\bar{x}$ direction and $R(r)$ is the linear eigenfunction of the mode excited in the fibre which depends on the coordinate perpendicular to $\bar{x}$. It

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can be shown [7 and references therein] that, if fibre loss and higher-order dispersion are negligible, the normalised envelope function Φ, denoted as u, satisfies the non-linear Schrödinger equation:

$$\mathrm{i}\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+|u|^2 u=0. \tag{2}$$

Here, t and x are the normalised coordinates $\bar{x}$ and $\bar{t}-\bar{x}/v_g$, respectively, and v_g is the group velocity.

To find out which type of initial envelope function generates solitons one does not have to solve the non-linear partial differential equation (2). All the information about the existence of solutions can be obtained from the eigenvalue problem [3]

$$A\boldsymbol{v}=\lambda\boldsymbol{v} \qquad \text{with} \qquad A=\begin{pmatrix} \mathrm{i}\,\mathrm{d}/\mathrm{d}x & u(x,0) \\ -u^*(x,0) & -\mathrm{i}\,\mathrm{d}/\mathrm{d}x \end{pmatrix}. \tag{3}$$

(Here and hereafter an asterisk denotes the complex conjugate.) Each discrete eigenvalue $\lambda=\kappa+\mathrm{i}\eta$ with L^2-integrable eigenfunction corresponds to a soliton with amplitude 2η moving with velocity 2κ. Equivalently, each one of the two second-order equations:

$$v_1''-\frac{u'}{u}v_1'+\left(\lambda^2-\mathrm{i}\lambda\frac{u'}{u}+|u|^2\right)v_1=0 \tag{4a}$$

$$v_2''-\frac{(u^*)'}{u^*}v_2'+\left(\lambda^2+\mathrm{i}\lambda\frac{(u^*)'}{u^*}+|u|^2\right)v_2=0 \tag{4b}$$

together with its corresponding first-order equation (3) contains all the information about solitons.

Since A is not a normal operator most of the standard theory does not apply. For this reason we solve the eigenvalue problem (3) for two special families of initial envelope functions explicitly. Both families consist of a purely imaginary $u(x,0)$ together with its Galilei transforms $u(x,0)\exp(-\mathrm{i}Vx)$ [6]. Because a Galilei transformation only shifts κ by $\frac{1}{2}V$, we can concentrate on $u(x,0)$ itself and determine its discrete eigenvalues. In fact, we concentrate on the purely imaginary eigenvalues of A. (In [3], it is claimed that these are all the eigenvalues of A in the case of purely imaginary $u(x,0)$.) Therefore, the solitons which we might find in the initial Galilei transformed pulse constitute a soliton pulse varying periodically in shape and propagating with velocity V [3, 6]. This pulse is called a bound state of solitons, which is the type of soliton pulse observed in experiments.

For purely imaginary $u(x,0)=-\mathrm{i}q(x)$, which falls off fast enough at $\pm\infty$, we want to find all continuous solutions of

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}-\eta\sigma_3-q\mathrm{i}\sigma_2\right)\boldsymbol{v}=0 \qquad \eta\in\boldsymbol{R},\ \eta>0 \tag{5}$$

with asymptotic behaviour

$$\exp(\eta x)\begin{pmatrix}1\\0\end{pmatrix}\xleftarrow[x\to-\infty]{}\begin{pmatrix}v_1\\v_2\end{pmatrix}\xrightarrow[x\to+\infty]{}c\exp(-\eta x)\begin{pmatrix}0\\1\end{pmatrix} \tag{6}$$

(σ_2 and σ_3 are Pauli matrices). After solving for positive and negative x, we will always be able to match the two parts of v_2, say, by choosing an appropriate value for c. Matching the two parts of v_1 means finding the eigenvalues η. If q is not large enough to stop v_1 from increasing and to make it decrease sufficiently fast there will not be any eigenvalue. If q is sufficiently large it will also be possible to produce nodes in v_1 and v_2 for larger eigenvalues. The following special cases will support this general discussion.

In both special cases we discuss, q is positive definite, i.e. $q(x) \geqslant 0$. (If $q(x) \leqslant 0$, the situation is completely analogous because (v_1, v_2) is an eigenfunction for $q(x)$ iff $(v_1, -v_2)$ is an eigenfunction for $-q(x)$ with the same eigenvalue.) In our first case, q is given by

$$q(x) = \begin{cases} 0 & \text{for} \quad |x| > \frac{1}{2}a \\ b & \text{for} \quad |x| \leqslant \frac{1}{2}a \end{cases} \qquad b > 0. \tag{7}$$

For this q, the solutions are

$$x > -\tfrac{1}{2}a: \qquad v_1 = \exp(\eta x) \qquad v_2 = 0 \tag{8a}$$

$$\begin{aligned} |x| \leqslant \tfrac{1}{2}a: \qquad & v_1 = c_1 \cos(b^2 - \eta^2)^{1/2} x + c_2 \sin(b^2 - \eta^2)^{1/2} x \\ & v_2 = (1/b)\{[c_2(b^2 - \eta^2)^{1/2} - c_1 \eta] \cos(b^2 - \eta^2)^{1/2} x \\ & \qquad - [c_1(b^2 - \eta^2)^{1/2} + c_2 \eta] \sin(b^2 - \eta^2)^{1/2} x\} \end{aligned} \tag{8b}$$

$$x > \tfrac{1}{2}a: \qquad v_1 = 0 \qquad v_2 = c \exp(-\eta x) \tag{8c}$$

with $\eta^2 < b^2$. It is easy to see that if $\eta^2 \geqslant b^2$, the solutions cannot be matched continuously at $x = \pm\frac{1}{2}a$.

For (8a)-(8b), the matching conditions at $x = \pm\frac{1}{2}a$ yield

$$\begin{aligned} & c_1 \cos(b^2 - \eta^2)^{1/2} \tfrac{1}{2} a = \tfrac{1}{2} \exp(-\eta \tfrac{1}{2} a) \\ & c_2 \sin(b^2 - \eta^2)^{1/2} \tfrac{1}{2} a = -\tfrac{1}{2} \exp(-\eta \tfrac{1}{2} a) \\ & cb \sin(b^2 - \eta^2)^{1/2} a = -(b^2 - \eta^2)^{1/2} \end{aligned} \tag{9}$$

and

$$\eta = -(b^2 - \eta^2)^{1/2} \cot(b^2 - \eta^2)^{1/2} a. \tag{10}$$

(Equation (10) implies $c = \pm 1$.) Our question about the soliton content of the input pulse has been reduced to the question about the solutions η of (10).

If we define $\rho = (b^2 - \eta^2)^{1/2} a$, η is a solution of (10) iff ρ and η satisfy

$$\rho^2 + a^2 \eta^2 = a^2 b^2 \tag{11a}$$

and

$$\eta = -(\rho / a) \cot \rho \tag{11b}$$

for $0 < \eta < b$ and $0 < \rho < ab$. The soliton number N is therefore

$$N = \langle \tfrac{1}{2} + ab/\pi \rangle = \langle \tfrac{1}{2} + F/\pi \rangle \tag{12}$$

where

$$F = \int_{-\infty}^{+\infty} |u(x, 0)| \, dx \tag{13}$$

and $\langle \ldots \rangle$ denotes the integer smaller than the argument. In terms of F, N is the same in this case as in the case of an envelope function of sech(x) form [6].

As a second special case we study (5) for

$$q(x) = \beta \exp(-\alpha|x|) \qquad \alpha, \beta > 0. \tag{14}$$

For this q, (4) can be written as Bessel's equation [8]:

$$\frac{d^2\psi}{ds^2} + \frac{1}{s}\frac{d\psi}{ds} + \left(1 - \frac{\nu^2}{s^2}\right)\psi = 0 \tag{15}$$

if we use the following variables:

$$x < 0: \qquad s = \frac{\beta}{\alpha}\exp(\alpha x) \qquad \psi = \frac{v_1}{\sqrt{s}} \tag{16a}$$

$$x > 0: \qquad s = \frac{\beta}{\alpha}\exp(-\alpha x) \qquad \psi = \frac{v_2}{\sqrt{s}}. \tag{16b}$$

For all x

$$\nu^2 = \left(\frac{\eta}{\alpha} - \frac{1}{2}\right)^2. \tag{17}$$

The solution of (15) with correct asymptotic behaviour at $s = 0$ is the Bessel function of order ν:

$$\begin{aligned} J_\nu(s) &= \sum_{m=0}^{\infty} \frac{(-1)^m(\frac{1}{2}s)^{\nu+2m}}{\Gamma(\nu+m+1)\Gamma(m+1)} \\ &= \frac{1}{\Gamma(\nu+1)}\left(\frac{s}{2}\right)^\nu \left[1 - \frac{1}{\nu+1}\left(\frac{s}{2}\right)^2 + \frac{1}{2(\nu+1)(\nu+2)}\left(\frac{s}{4}\right)^4 - \ldots\right] \end{aligned} \tag{18}$$

with $\nu = \eta/\alpha - \frac{1}{2} > -\frac{1}{2}$. Hence

$$\begin{aligned} x < 0: \qquad & v_1 = 2^\nu \left(\frac{\alpha}{\beta}\right)^{\eta/\alpha} \Gamma(\nu+1)\sqrt{s}J_\nu(s) \\ & v_2 = -2^\nu \left(\frac{\alpha}{\beta}\right)^{\eta/\alpha} \Gamma(\nu+1)\sqrt{s}J_{\nu+1}(s) \end{aligned} \tag{19a}$$

$$\begin{aligned} x > 0: \qquad & v_2 = c2^\nu \left(\frac{\alpha}{\beta}\right)^{\eta/\alpha} \Gamma(\nu+1)\sqrt{s}J_\nu(s) \\ & v_1 = -c2^\nu \left(\frac{\alpha}{\beta}\right)^{\eta/\alpha} \Gamma(\nu+1)\sqrt{s}J_{\nu+1}(s) \end{aligned} \tag{19b}$$

are the solutions v_1 and v_2.

Continuity of v_1 and v_2 at $x = 0$ implies $c = \mp 1$ and

$$J_\nu(\beta/\alpha) = \pm J_{\nu+1}(\beta/\alpha) \qquad \nu > -\tfrac{1}{2}. \tag{20}$$

This is the eigenvalue relation for $q(x)$ given in (14). η is an eigenvalue if J_ν intersects $J_{\nu+1}$ or $-J_{\nu+1}$ at β/α, where $\nu = \eta/\alpha - \frac{1}{2}$. We therefore have to study the points of intersection of J_ν with $\pm J_{\nu+1}$.

Let s_n^ν, $\nu = -\frac{1}{2}$, denote the coordinate of the nth point of intersection of $J_{-1/2}$ and $\pm J_{1/2}$. Since

$$J_{-1/2} = \left(\frac{2}{\pi}\right)^{1/2} \frac{\cos(s)}{\sqrt{s}} \qquad \text{and} \qquad J_{1/2} = \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin(s)}{\sqrt{s}} \tag{21}$$

$s_n^{-1/2} = (2n-1)\frac{1}{4}\pi$ holds. If, starting from $\nu = -\frac{1}{2}$, we increase ν then the points of intersection will change continuously with ν. This follows from (18). Furthermore, while increasing ν, no pair of intersections will move together to coincide and no new additional intersections will develop. The first statement must be true because the zeros of J_ν and $J_{\nu+1}$ separate each other. If the second statement were false, an intersection would have to split into at least three intersections for some $\nu = \nu_0$ at some $s = s_n^{\nu_0}$. This would imply that J_{ν_0} and $\pm J_{\nu_0+1}$ and their first two derivatives are equal at $s_n^{\nu_0}$ $(\nu_0 > -\frac{1}{2})$, which contradicts Bessel's equation.

So far we have seen that s_n^ν, which is continuous in ν, labels the points of intersection of J_ν with $\pm J_{\nu+1}$. That $s_n^\nu \to \infty$ for $n \to \infty$ follows from the fact that J_ν has arbitrarily large zeros. Also $s_n^\nu \to \infty$ for $\nu \to \infty$. This can be seen as follows. The first term in (18) is the dominant one if $s < \nu^{1/3}$ and if ν is large. However, $(s/2)^\nu/\Gamma(\nu+1)$ and $(s/2)^{\nu+1}/\Gamma(\nu+2)$ do not intersect for any $s < \nu^{1/3}$. (In fact, they intersect at $s = 2(\nu+1)$.)

Finally, we show that s_n^ν is monotonically increasing with ν. Equation (5) implies

$$\int_{-\infty}^{+\infty} \left(\frac{\mathrm{d}}{\mathrm{d}x} \boldsymbol{v}^{\mathrm{T}}\sigma_2 + \boldsymbol{v}^{\mathrm{T}}\sigma_2(\eta\sigma_3 + \mathrm{i}q\sigma_2)\right)\boldsymbol{w}\, \mathrm{d}x = 0 \tag{22}$$

for any $\boldsymbol{w}$. If we change q by a small amount δq and thus $\boldsymbol{v}$ by $\boldsymbol{\delta v}$ and η by $\delta\eta$, we obtain

$$-2\delta\eta \int_{-\infty}^{+\infty} v_1 v_2\, \mathrm{d}x = \int_{-\infty}^{+\infty} \delta q \boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}\, \mathrm{d}x \tag{23}$$

to first order, using (22). Since

$$\int_{-\infty}^{+\infty} \delta q \boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}\, \mathrm{d}x > 0 \tag{24}$$

for positive definite δq and

$$-\int_{-\infty}^{+\infty} v_1 v_2\, \mathrm{d}x = \frac{1}{\alpha} 2^{2\nu+1} \left(\frac{\alpha}{\beta}\right)^{2\nu/\alpha} \Gamma^2(\nu+1) \int_0^1 J_\nu(s) J_{\nu+1}(s)\, \mathrm{d}s > 0 \tag{25}$$

ν increases if β increases. (That the inequality (25) holds can be deduced from the fact that the first zero of $J_{-1/2}$ is at $\frac{1}{2}\pi > 1$ and that the first zero of J_ν increases with ν.) This rules out the possibility of s_n^ν decreasing at any ν. s_n^ν also cannot stay constant for any range of ν. If it did, $\delta\eta \neq 0$ and $\delta q = 0$ would be possible, which again contradicts (23).

Our analysis shows that the soliton number N is equal to the number of intersections of $J_{-1/2}$ with $\pm J_{1/2}$ below β/α, which is the integer smaller than $\frac{1}{2} + 2\beta/\alpha\pi$. Since (13) yields $F = 2\beta/\alpha$, again

$$N = \langle \tfrac{1}{2} + F/\pi \rangle \tag{26}$$

holds. To understand our results better we intend to extend our analysis to the 'non-soliton' part of the pulse. Our main aim, however, is to use the experience gained from studying our two examples and the sech(x) pulse to discuss 'realistic' pulses from 'realistic' sources.

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